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CONTACT PROBLEMS OF THE THEORY OF ELASTICITY WITH FRICTION AND ADHESION†

YU. A. ANTIPOV and N. KH. ARUTYUNYAN

Odessa

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An exact closed solution of the plane contact problem for a semi-infinite stamp is constructed for the case when the free boundary of the half-plane is under a load (problem 1), or for an analytic solution, to any prescribed accuracy, of the problem of a finite stamp impressed into an elastic half-plane under the action of a central vertical force P (problem 2), or under the action of the above force P, a horizontal force T and a pair of forces with moment M (problem 3). In all three cases the region of contact consists of a zone of adhesion and fraction, and the stamp has a plane profile.

THE USE of the zones of Coulomb friction in formulating the problem of impressing a stamp with incomplete adhesion into the medium was proposed in [1]. The conditions for the removal of stress singularities from the neighbourhood of the point of transition from the zone of slippage (taking kinetic friction into account) to separation into layers and from the adhesion of the slippage zone, were studied in [2] for the plane problem of a composite elastic plane. In order to enable the Irwin criterion to be applied to the problem of the delamination of heterogeneous materials, a segment was isolated in [3], within the slippage zone, on which shear stresses were specified.

An approximate solution of problem 2 was constructed in [1] with the help of conformal mapping. The method was extended in [4] to the solution of the problem of impressing a stamp under the action of an eccentrically applied force. Below a different approach is proposed, based on reducing the above problems to a Riemann vector problem for one, two and three pairs of functions (for problem 1, 2 and 3, respectively), which are then solved using the method given in [5]. The boundaries of the adhesion and friction zones not known in advance are found, and formulas are derived for the contact stresses with explicitly isolated power singularities. Numerical examples are given.

1. THE PROBLEM OF A SEMI-INFINITE STAMP

Let us consider an elastic half-plane ($0<r<\infty$, $-\pi<\theta<0$), whose Poisson's ratio is ν and whose modulus of elasticity is E. Stresses $\sigma_\theta=f_1(r)$, $\tau_{r\theta}=f_2(r)$ are applied to part of the boundary of the half-plane ($0<r<\infty$, $\theta=-\pi$). When $\theta=0$, the half-plane is in contact with a semi-infinite stamp. The region of contact is separated into the adhesion zone ($b<r<\infty$): $u_\theta=\delta_n$, $u_r=\delta_t$ (δ_n, δ_t are the normal and tangential displacements of the stamp), and the Coulomb fiction zone ($0<r<\infty$):

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$\tau_{r\theta}-\mu\sigma_\theta=0$, $u_\theta=\delta_n$ (μ is the coefficient of friction). In the adhesion part the shear stresses are insufficient to cause slippage $\tau_{r\theta}-\mu\sigma_\theta<0$ ($b<r<\infty$, $\theta=0$). The normal contact pressure under the stamp must be positive everywhere $\sigma_\theta>0$ ($0<r<\infty$, $\theta=0$).

We will introduce a pair of unknown functions

$$\chi(r)=(\tau_{r\theta}-\mu\sigma_\theta)_{\theta=0},\quad \psi(r)=(1+\nu)^{-1}E\partial u_r/\partial r\,|_{\theta=0},\quad 0<r<\infty$$

with $\operatorname{supp}\chi(r)\subset[b,\infty)$, $\operatorname{supp}\psi(r)\subset[0,b]$, and apply the Mellin transformation

$$\|\sigma_{\theta s},\tau_{r\theta s},\eta_{rs},\eta_{\theta s}\|=\int_0^\infty\left\|\sigma_\theta,\tau_{r\theta},\frac{\partial u_r}{\partial r},\frac{\partial u_\theta}{\partial r}\right\|r^s\,dr \tag{1.1}$$

to the equations of statics, the conditions of continuity and the physical equations. Then (the case of plane strain)

$$\sigma_{\theta s}^{IV}+2(s^2+1)\sigma_{\theta s}''+(s^2-1)^2\sigma_{\theta s}=0,\quad -\pi<\theta<0, \tag{1.2}$$

$$(s-1)\tau_{r\theta s}=\sigma_{\theta s}',\quad s(s-1)\eta_{rs}=\nu_*[(1-\nu)\sigma_{\theta s}''+$$
$$+(\nu s+1-\nu)(s-1)\sigma_{\theta s}]$$
$$s(s^2-1)\eta_{\theta s}=\nu_*[(1-\nu)\sigma_{\theta s}'''+(2s^2-\nu s^2+2\nu s-s+1-\nu)\sigma_{\theta s}']$$
$$(\nu_*=(1+\nu)E^{-1}).$$

Requiring that the solution of Eq. (1.2) must satisfy the boundary conditions

$$[(s-1)^{-1}\sigma_{\theta s}'-\mu\sigma_{\theta s}]_{\theta=0}=b^{s+1}\Phi^+(s)$$
$$[(1-\nu)\sigma_{\theta s}''-(\nu s+1-\nu)(s-1)\sigma_{\theta s}]_{\theta=0}=s(s-1)b^{s+1}\Phi^-(s)$$
$$[(1-\nu)\sigma_{\theta s}'''+(2s^2-\nu s^2+2\nu s-s+1-\nu)\sigma_{\theta s}']_{\theta=0}=0$$
$$\sigma_{\theta s}|_{\theta=-\pi}=f_{1s},\quad \sigma_{\theta s}'|_{\theta=-\pi}=(s-1)f_{2s}$$
$$\Phi^+(s)=\int_1^\infty\chi(br)r^s\,dr,\quad \Phi^-(s)=\int_0^1\psi(br)r^s\,dr,\quad f_{js}=\int_0^\infty f_j(r)r^s\,dr$$

we arrive at the Riemann problem

$$\Phi^+(s)=G(s)\Phi^-(s)+g(s),\quad s\in\Gamma:\ \operatorname{Re}(s)=\gamma_0 \tag{1.3}$$
$$G(s)=4\delta_0^{-1}(s)\sin\pi s(\varkappa_+\cos\pi s-\mu\varkappa_-\sin\pi s),\quad \varkappa_\pm={}^1/_2(\varkappa\pm1)$$

$$g(s)=4\varkappa_+b^{-s-1}\delta_0^{-1}(s)\sin\pi s[(\varkappa_-+\mu\varkappa_+\operatorname{ctg}\pi s)f_{1s}+(\mu\varkappa_--\varkappa_+\operatorname{ctg}\pi s)f_{2s}] \tag{1.4}$$
$$\varkappa=3-4\nu,\quad \delta_0(s)=1+2\varkappa\cos2\pi s+\varkappa^2$$

for the pair of functions $\Phi^\pm(s)$ analytic in $D^\pm$: $\operatorname{Re}(s)\gtrless\gamma_0\in(-\varepsilon,0)$ $(0<\varepsilon<1)$. Let us write the function $G(s)$ in the form

$$G(s)=\varkappa_\alpha\sin\pi s\sin\pi(\alpha-s)\sec\pi(s+i\beta)\sec\pi(s-i\beta)=K^+(s)K^-(s)$$
$$\varkappa_\alpha=\varkappa_+(\varkappa\sin\pi\alpha)^{-1},\quad \alpha=\pi^{-1}\operatorname{arcctg}(\mu\varkappa_-\varkappa_+^{-1}),\quad \beta=(2\pi)^{-1}\ln\varkappa$$
$$K^+(s)=\frac{\Gamma({}^1/_2-s-i\beta)\Gamma({}^1/_2-s+i\beta)}{\Gamma(-s)\Gamma(\alpha-s)} \tag{1.5}$$
$$K^-(s)=-\frac{\varkappa_\alpha\Gamma({}^1/_2+s+i\beta)\Gamma({}^1/_2+s-i\beta)}{\Gamma(1+s)\Gamma(1-\alpha+s)}$$

The functions $K^{\pm}(s)$ are analytic and do not vanish in $D^{\pm}$.
The solution of problem (1.3) has the form

$$\Phi^{+}(s) = K^{+}(s)\,\Psi^{+}(s), \quad \Phi^{-}(s) = [K^{-}(s)]^{-1}\,\Psi^{-}(s)$$

$$\Psi(s) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)}{K^{+}(t)}\frac{dt}{t-s}$$

Using the inverse Mellin transformation we obtain

$$\psi(r) = \frac{1}{2\pi i}\int\limits_{\Gamma}[K^{+}(s)\,\Psi^{+}(s) - g(s)]\frac{1}{G(s)}\left(\frac{r}{b}\right)^{-s-1}ds, \quad 0<r<b$$

$$\chi(r) = \frac{1}{2\pi i}\int\limits_{\Gamma}K^{+}(s)\,\Psi^{+}(s)\left(\frac{r}{b}\right)^{-s-1}ds, \quad r>b \tag{1.6}$$

$$\sigma_{\theta}|_{\theta=0} = \frac{1}{2\pi i}\int\limits_{\Gamma}(\varkappa_{-}\sin^{2}\pi s b^{s+1}\Phi^{-}(s) - \varkappa_{+}^{2}\cos\pi s f_{1s} - \varkappa_{+}\varkappa_{-}\sin\pi s f_{2s})\frac{4r^{-s-1}}{\delta_{0}(s)}\,ds$$

$$(0<r<\infty)$$

Applying the method of residues and a Tauber-type theorem to relations (1.6), we obtain

$$\left.\frac{\partial u_r}{\partial r}\right|_{\theta=0} = O(r^{-\alpha}), \quad \sigma_{\theta}|_{\theta=0} = O(r^{-\alpha}), \quad r\to 0$$

$$\left.\frac{\partial u_r}{\partial r}\right|_{\theta=0} = \frac{\nu_{*}\Psi_{0}(b)}{\varkappa_{\alpha}\Gamma(\alpha)}\left(1-\frac{r}{b}\right)^{\alpha-1}, \quad r\to b-0; \quad \Psi_{0}(b) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(s)\,ds}{K^{+}(s)}$$

$$\chi(r) \sim \Psi_{0}(b)\,[\Gamma(\alpha)]^{-1}(r/b-1)^{\alpha-1}, \quad r\to b+0 \tag{1.7}$$

In order to solve the problem completely, we must find the position of the point b, unknown *a priori*. Let us introduce the stress intensity factor

$$K(b) = \lim_{r\to b+0}(r-b)^{1-\alpha}(\tau_{r\theta} - \mu\sigma_{\theta})_{\theta=0}$$

and require that $K(b)=0$, i.e. that $\tau_{r\theta}-\mu\sigma_{\theta}=0$, $r=b$, $\theta=0$. Then, as a result of (1.7), the position of point b will be given by the condition

$$\Psi_{0}(b) = 0 \tag{1.8}$$

Here the contact stresses $\tau_{r\theta}$ and σ_{θ} will remain bounded in the neighbourhood of the point $r=b$. We note that (1.8) is equivalent to condition [1] for determining the length of the adhesion zone. In [2] the boundedness of the stresses at the point of transition from the Coulomb friction zone to adhesion is obtained in a different manner for the case of the problem of the contact of a half-plane with a single point at which the boundary conditions change.

Let us evaluate the integral (1.7) and obtain a numerical formula for the contact stress σ_{θ} in the case of the Flamant problem (Fig. 1). Let $f_1(r) = -P\delta(r-c)$, $f_2(r)=0$ ($P=$ const). Then

$$f_{1s} = -Pc^{s}, \; f_{2s} = 0, \; g(s) = -4\varkappa_{+}\delta_{0}^{-1}(s)\,Pb^{-1}\lambda^{-s}\zeta(s), \; \lambda = b/c$$

$$\zeta(s) = \mu\varkappa_{+}\cos\pi s + \varkappa_{-}\sin\pi s, \; \Psi_{0}(b) = -\varkappa_{+}(\pi\varkappa b)^{-1}P\omega(\lambda)$$

$$\pi\omega(\lambda) = \frac{1}{2\pi i}\int\limits_{\Gamma}\Gamma(-s)\,\Gamma(\alpha-s)\,\Gamma\left(\frac{1}{2}+s+i\beta\right)\Gamma\left(\frac{1}{2}+s-i\beta\right)\zeta(s)\,\lambda^{-s}ds$$

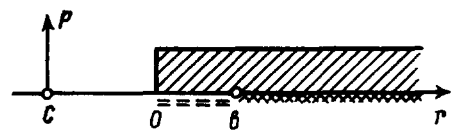

FIG. 1.

We will evaluate the last integral using the theory of residues. We obtain

$$\begin{gathered}
\omega(\lambda) = (\pi\lambda\varkappa)^{1/2}\,\mathrm{Re}\,\{(\mu - i)(\lambda/4)^{i\beta}\Gamma(z)\,[\Gamma(1+i\beta)]^{-1} \times \\
\times F(z_0, z; 2z_0; -\lambda),\ 0<\lambda<1 \\
\omega(\lambda) = \mu\varkappa^{1/2}\Gamma(\alpha)\,F(z_0, \bar z_0; 1-\alpha; -\lambda^{-1}) - \varkappa_-(\mu^2+1) \times \\
\times [\lambda^{\alpha}\Gamma(1+\alpha)]^{-1}\,|\Gamma(z)|^2 F(z, \bar z; 1+\alpha; -\lambda^{-1}),\ \lambda>1 \\
z = \alpha + 1/2 + i\beta,\ z_0 = 1/2 + i\beta
\end{gathered}$$

The quantity λ is found from the equation $\omega(\lambda) = 0$. Evaluating the integral (1.6) and considering the case of $0<r<b$ and $r>b$, $0<\lambda<1$ as well as $\lambda>1$, we arrive at the following computational formulas for the contact stress:

$$\begin{gathered}
\sigma_\theta|_{\theta=0} = \frac{\varkappa_-\sin^2\pi\alpha}{b\pi^2\varkappa^{1/2}} P \sum_{j=0}^{\infty} h_j^{(1)}(r) \sum_{m=0}^{\infty} g_m^{(1)}\,\mathrm{Re}\left(\frac{d_m^{(0)}}{-m+j+1-z}\right), \quad r<b, \quad \lambda<1 \\
\sigma_\theta|_{\theta=0} = \frac{\sin\pi\alpha}{\pi} P\left\{\frac{(r/c)^{-\alpha}}{r+c} + \frac{\varkappa_-}{b\pi\varkappa}\sum_{j=0}^{\infty} h_j^{(1)}(r)\sum_{m=0}^{\infty} g_m^{(2)} \times \right. \\
\left. \times\left(\frac{d_m^{(1)}}{m-\alpha+j+1} - \frac{d_m^{(2)}}{m+j+1}\right)\right\}, \quad r<b, \quad \lambda>1 \\
\sigma_\theta|_{\theta=0} = \frac{\varkappa_+ P}{\pi\varkappa^{1/2}}\left\{\frac{\cos(\beta\ln(r/c))}{r+c}\left(\frac{c}{r}\right)^{1/2} + \frac{\sin\pi\alpha}{2\pi b}\,\mathrm{Im}\left[\sum_{j=0}^{\infty} h_j^{(2)}(r)\sum_{m=0}^{\infty} g_m^{(1)} \times \right.\right. \\
\left.\left. \times\left(\frac{d_m^{(0)}}{m+j+2z_0} + \frac{\overline{d_m^{(0)}}}{m+j+1}\right)\right]\right\}, \quad r>b, \quad \lambda<1 \\
\sigma_\theta|_{\theta=0} = -\frac{\varkappa_+ P}{\pi^2 b\varkappa}\,\mathrm{Im}\left\{\sum_{j=0}^{\infty} h_j^{(2)}(r)\sum_{m=0}^{\infty} g_m^{(2)}\left(\frac{d_m^{(1)}}{m-j-z_0} - \frac{d_m^{(2)}}{m-j+\bar z-1}\right)\right\} \\
r>b,\ \lambda>1
\end{gathered} \tag{1.9}$$

Here

$$\begin{gathered}
h_j^{(1)}(r) = \frac{|\Gamma(2-\bar z+j)|^2}{\Gamma(1-\alpha+j)\,j!}\left(\frac{r}{b}\right)^{j-\alpha}, \quad h_j^{(2)}(r) = \frac{\Gamma(1+z_0+j)\,\Gamma(2-\bar z+j)}{\Gamma(j+2z_0)\,j!}\left(\frac{r}{b}\right)^{-3/2-j-i\beta} \\
d_m^{(0)} = (\mu-i)\,\lambda^{i\beta}\frac{\Gamma(m+z_0)\,\Gamma(m+z)}{\Gamma(m+2z_0)}, \quad d_m^{(1)} = \mu\varkappa_+\frac{|\Gamma(m+z_0)|^2}{\Gamma(1-\alpha+m)} \\
d_m^{(2)} = \frac{\mu^2+1}{\lambda^{\alpha}}\varkappa_-\sin\pi\alpha\frac{|\Gamma(m+z)|^2}{\Gamma(1+\alpha+m)}; \quad g_m^{(1)} = \frac{(-1)^m\lambda^{m+1/2}}{m!}, \quad g_m^{(2)} = \frac{(-1)^m}{\lambda^m m!}
\end{gathered}$$

Formulas (1.9) yield the contact stress intensity factor

$$K_0\lambda = \lim_{r\to 0} r^{\alpha}\sigma_\theta(r, 0)$$

$$K_0(\lambda) = \frac{\varkappa_- \lambda^{1/2} P \sin^2 \pi\alpha b^{\alpha-1}}{(\pi\varkappa)^{1/2}\Gamma(1-\alpha)} \operatorname{Re}\left\{(\mu - i)\left(\frac{\lambda}{4}\right)^{i\beta} \frac{\Gamma(2-\bar{z})}{\Gamma(1+i\beta)\sin \pi z} \times \right.$$
$$\left. \times F(z_0, z-1; 2z_0; -\lambda)\right\}, \quad \lambda < 1$$

$$K_0(\lambda) = \frac{P}{\pi} \sin \pi\alpha c^{\alpha-1}\left\{\frac{\varkappa_- \lambda^{\alpha-1}\mu\,|\,\Gamma(2-\bar{z})\,|^2}{\varkappa^{1/2}\Gamma(1-\alpha)\Gamma(2-\alpha)} F\left(z_0, \bar{z}_0; 2-\alpha; -\frac{1}{\lambda}\right) + \right.$$
$$\left. + F(z-1, \bar{z}-1; \alpha; -\lambda^{-1})\right\}, \quad \lambda > 1$$

2. L. A. GALIN'S PROBLEM (PROBLEM 2)

Let a stamp ($0<r<a$, $\theta = -\pi$; $0<r<a$. $\theta = 0$) be impressed into the elastic half-plane ($0<r<\infty$, $-\pi<\theta<0$) under the action of a vertical force P applied at the point $r = 0$. The region of contact is split into the adhesion zone $u_\theta = \delta_n$, $u_r = \delta_t$ ($0<r<b$, $\theta = -\pi$;$0<r<b$, $\theta = 0$), and the Coulomb friction zone $u_\theta = \delta_n$, $\tau_{r\theta} - \mu\sigma_\theta = 0$ ($b<r<a$, $\theta = -\pi$); $u_\theta = \delta_n$, $\tau_{r\theta} + \mu\sigma_\theta = 0$ ($b<r<a$, $\theta = 0$). Outside the zone of contact the boundary of the half-plane is load-free. As in Sec. 1, the normal stresses must be positive within the region of contact, and shear stresses in the adhesion zone satisfy the inequality $|\tau_{r\theta}|<\mu|\sigma_\theta|$.

Taking symmetry into account, we can reduce the problem formulated here to a problem for a quarter-plane:

$$\begin{gathered} u_r|_{\theta=0} = \delta_t, \quad 0<r<b; \quad (\tau_{r\theta} + \mu\sigma_\theta)_{\theta=0} = 0, \quad b<r<a \\ u_\theta|_{\theta=0} = \delta_n, \quad 0<r<a; \quad \sigma_\theta|_{\theta=0} = \tau_{r\theta}|_{\theta=0} = 0; \quad r>a \\ u_\theta|_{\theta=-\pi/2} = \tau_{r\theta}|_{\theta=-\pi/2} = 0, \quad 0<r<\infty \end{gathered} \tag{2.1}$$

with the additional condition of equilibrium of the stamp

$$\int_0^a \sigma_\theta|_{\theta=0}\, dr = \frac{P}{2} \tag{2.2}$$

Let us introduce into our discussion the unknown functions

$$\begin{gathered} \chi_1(r) = \sigma_\theta(r, 0), \quad \chi_2(r) = \tau_{r\theta}(r, 0) + \mu\sigma_\theta(r, 0) \\ \psi_1(r) = \nu_*^{-1}\partial u_r/\partial r(r, 0), \quad \psi_2(r) = \nu_*^{-1}\,\partial u_\theta/\partial r(r, 0) \end{gathered} \tag{2.3}$$

Then from the boundary conditions (2.1) we have $\operatorname{supp}\chi_1 \subset [0, a]$, $\operatorname{supp}\chi_2 \subset [0, b]$, $\operatorname{supp}\psi_1 \subset [b, \infty]$, $\operatorname{supp}\psi_2 \subset [a, \infty]$, and the function $\chi_1(r)$, by virtue of (2.2), must satisfy the condition

$$\int_0^a \chi_1(r)\, dr = \frac{P}{2} \tag{2.4}$$

Using the Mellin transformation (1.2) we reduce the problem formulated here to the following boundary value problem for Eq. (1.2):

$$\sigma_{\theta s}\,|_{\theta=0} = a^{s+1}\Phi_1^-(s), \quad [\mu\sigma_{\theta s} + (s-1)^{-1}\sigma_{\theta s}']_{\theta=0} = b^{s+1}\Phi_2^-(s)$$
$$[(1-\nu)\,\sigma_{\theta s}'' - (\nu s + 1 - \nu)(s-1)\,\sigma_{\theta s}]_{\theta=0} = s(s-1)\,b^{s+1}\Phi_1^+(s)$$
$$[(1-\nu)\,\sigma_{\theta s}''' + (2s^2 - \nu s^2 + 2\nu s - s + 1 - \nu)\,\sigma_{\theta s}']_{\theta=0} =$$
$$= s(s^2-1)\,a^{s+1}\Phi_2^+(s)$$
$$\sigma_{\theta s}'\,|_{\theta=-\pi/2} = \sigma_{\theta s}'''\,|_{\theta=-\pi/2} = 0$$
$$\Phi_1^-(s) = \int_0^1 \chi_1(ar)\,r^s\,dr, \quad \Phi_2^-(s) = \int_0^1 \chi_2(br)\,r^s\,dr$$
$$\Phi_1^+(s) = \int_1^\infty \psi_1(br)\,r^s\,dr, \quad \Phi_2^+(s) = \int_1^\infty \psi_2(ar)\,r^s\,dr \tag{2.5}$$

Solving this problem we obtain the homogeneous Riemann problem

$$\lambda^{s+1}\Phi_1^+(s) = K_1(s)\,\Phi_1^-(s) - \varkappa_+\lambda^{s+1}\,\mathrm{tg}\,{}^1/_2\pi s\Phi_2^-(s)$$
$$\Phi_2^+(s) = K_0(s)\,\Phi_1^-(s) - \varkappa_-\lambda^{s+1}\Phi_2^-(s), \quad s \in \Gamma \tag{2.6}$$
$$K_0(s) = \varkappa_+\,\mathrm{ctg}\,{}^1/_2\pi s + \mu\varkappa_-, \quad K_1(s) = \varkappa_- + \mu\varkappa_+\,\mathrm{tg}\,{}^1/_2\pi s,$$
$$\lambda = b/a \in (0, 1)$$

[the quantities $\varkappa_\pm$ are defined in (1.4)]. Deriving the expression for $\Phi_1^-(s)$ from the second equation of (2.6) and substituting it into the first equation, we obtain

$$\Phi_1^+(s) = [K_0(s)]^{-1}K_1(s)\,\lambda^{-s-1}\Phi_2^+(s) + \varkappa_*[K_0(s)]^{-1}\Phi_2^-(s),$$
$$\varkappa_* = \varkappa_-^2 - \varkappa_+^2$$

We now factorize the function $K_0(s) = K_0^+(s)K_0^-(s)$

$$K_0^+(s) = -\frac{\varkappa_0\Gamma(-s/2)}{\Gamma(1-\alpha-s/2)}, \quad K_0^-(s) = \frac{\Gamma(1+s/2)}{\Gamma(\alpha+s/2)}, \quad \varkappa_0 = \frac{\varkappa_+}{\sin\pi\alpha} \tag{2.7}$$

[α is defined in (1.5)], and rewrite system (2.6) in a form suitable to use the method of [5]

$$\varkappa_*[K_0^-(s)]^{-1}\Phi_2^-(s) = K_0^+(s)\,\Phi_1^+(s) - \lambda^{-s-1}[K_0^-(s)]^{-1}K_1(s)\,\Phi_2^+(s)$$
$$[K_0^+(s)]^{-1}\Phi_2^+(s) = K_0^-(s)\,\Phi_1^-(s) - \varkappa_-\lambda^{s+1}[K_0^+(s)]^{-1}\Phi_2^-(s) \tag{2.8}$$

The function $[K_0^-(s)]^{-1}K_1(s)$ is meromorphic in the region D^+ and has poles at the points $s=-2\alpha-2j$ and $s=-1-2j$, ($j=0, 1, \ldots$), while the function $[K_0^+(s)]^{-1}$ has no other singularities in the region D^- apart from the poles at the points $s=-2\alpha+2+2j$ ($j=0, 1, \ldots$). Let us introduce the functions

$$\Psi_0^\pm(s) = \sum_{j=0}^\infty \frac{A_j^\pm}{s+2\alpha\mp 2j\mp 1-1}, \quad \Psi_1^-(s) = \sum_{j=0}^\infty \frac{B_j}{s+1+2j} \tag{2.9}$$

$$A_j^+ = \operatorname*{Res}_{s=-2\alpha+2+2j}\{-\varkappa_-\lambda^{s+1}[K_0^+(s)]^{-1}\Phi_2^-(s)\}$$
$$A_j^- = \operatorname*{Res}_{s=-2\alpha-2j}\{-\lambda^{-s-1}[K_0^-(s)]^{-1}K_1(s)\,\Phi_2^+(s)\}$$
$$B_j = \operatorname*{Res}_{s=-1-2j}\{-\lambda^{-s-1}[K_0^-(s)]^{-1}K_1(s)\,\Phi_2^+(s)\} \tag{2.10}$$

Taking into account formulas (2.5) we have, according to Abel-type theorems, $\Phi_2^-(s) = O(s^{-\alpha})$, $\Phi_2^+(s) = O(s^{-1+\alpha})$, $s \to \infty$ when $s \in D^-$, D^+ respectively, in which case we obtain from (2.10) $A_j^+ = O(\lambda^{2j} j^{1-2\alpha})$, $A_j^- = O(\lambda^{2j} j^{-2+2\alpha})$, $B_j = O(\lambda^{2j} j^{-2+2\alpha})$, $j \to \infty$. Therefore the series (2.9) converge uniformly in the corresponding regions $D_0^\pm = \mathbf{C} \backslash UM_j^\mp$ $(j = 0, 1, \ldots)$, $D_1^- = \mathbf{C} \backslash UL_j^+$ $(j = 0, 1, \ldots)$ where $\mathbf{C}$ is the plane of the complex variable, $M_j^\mp = \{s \in \mathbf{C}: |s + 2\alpha \mp 2j \mp 1 - 1| < \varepsilon\}$, $L_j^+ = \{s \in \mathbf{C}: |s + 1 + 2j| < \varepsilon\}$, ε is a positive number as small as desired. Thus the function $\Psi_0^+(s)$ is analytic in D_0^+ and the function $\Psi_0^-(s)$, $\Psi_1^-(s)$ is analytic in D_0^-, D_1^- (we note that $D_0^\pm \supset D^\pm$, $D_1^- \supset D^-$).

Subtracting from the left- and right-hand sides of the first equation of (2.8) the sum $\Psi_0^-(s) + \Psi_1^-(s)$, and of the second equation the function $\Psi_0^+(s)$, amounts to removal of the poles and thus enables us to use the principle of continuity and Liouville's theorem. The formulas determining the solution of the Riemann problem (2.6) have the form

$$\Phi_1^-(s) = \frac{C + \Psi_0^+(s)}{K_0^-(s)} + \varkappa_- \lambda^{s+1} \frac{\Psi_0^-(s) + \Psi_1^-(s)}{\varkappa_* K_0^+(s)},$$

$$\Phi_1^+(s) = \frac{\Psi_0^-(s) + \Psi_1^-(s)}{K_0^+(s)} + \frac{K_1(s)\,[C + \Psi_0^+(s)]}{\lambda^{s+1} K_0^-(s)}$$

$$\Phi_2^-(s) = \varkappa_*^{-1} K_0^-(s)\,[\Psi_0^-(s) + \Psi_1^-(s)], \quad \Phi_2^+(s) = K_0^+(s)\,[C + \Psi_0^+(s)] \tag{2.11}$$

where C is an arbitrary constant. The unknown coefficients $A_j^\pm$, B_j are found from conditions (2.10). Substituting formulas (2.11), (2.7) and (2.9) into (2.10), we arrive at the following infinite, normal-type algebraic system:

$$A_{n*}^- = \lambda^{2\alpha - 1 + 2n} \delta_{0n}^+ \Big(1 - \sum_{j=0}^{\infty} \frac{A_{j*}^+}{2(n + j + 1)}\Big) \tag{2.12}$$

$$B_{n*} = \lambda^{2n} \delta_{1n}^+ \Big(1 - \sum_{j=0}^{\infty} \frac{A_{j*}^+}{2(n + j - \alpha) + 3}\Big)$$

$$A_{n*}^+ = \lambda^{2n + 3 - 2\alpha} \delta_{0n}^- \Big(\sum_{j=0}^{\infty} \frac{A_{j*}^-}{2(n + j + 1)} + \sum_{j=0}^{\infty} \frac{B_{j*}}{2(n + j - \alpha) + 3}\Big)$$

$$\delta_{0n}^+ = 2\varkappa_* \varkappa_+ \Gamma^2 (n + \alpha)\, (\pi \varkappa_- n!^2)^{-1}$$

$$\delta_{1n}^+ = 2\varkappa_+ \varkappa_0^2 \Gamma^2 (n + {}^1/_2)\, [\pi \varkappa_- \Gamma^2 (n + {}^3/_2 - \alpha)]^{-1}$$

$$\delta_{0n}^- = 2\varkappa_+ \varkappa_- \Gamma^2 (n + 2 - \alpha)\, (\pi \varkappa_* \varkappa_0^2 n!^2)^{-1}$$

for the new variables $A_{n*}^\pm$, B_{n*} connected with the old variables as follows: $A_n^\pm = CA_{n*}^\pm$, $B_n = CB_{n*}$.

We will solve system (2.12) by the asymptotic method. We will seek the coefficients $A_{n*}^\pm$, B_{n*} in the form of

$$A_{n*}^- = \lambda^{2n - 1 + 2\alpha} \sum_{j=0}^{\infty} a_{nj}^- \lambda^{2j}, \quad A_{n*}^+ = \lambda^{2n + 2} \sum_{j=0}^{\infty} a_{nj}^+ \lambda^{2j}$$

$$B_{n*} = \lambda^{2n} \sum_{j=0}^{\infty} b_{nj} \lambda^{2j} \tag{2.13}$$

Substituting the expansions (2.13) into system (2.12), we obtain the following recurrence relations:

$$
\begin{gathered}
a_{n0}^- = \delta_{0n}^+, \quad b_{n0} = \delta_{1n}^+ \\
a_{nk}^- = -\delta_{0n}^+ \sum_{j=1}^{k} \frac{a_{j-1,k-j}^+}{2(n+j)}, \quad b_{nk} = -\delta_{1n}^+ \sum_{j=1}^{k} \frac{a_{j-1,k-j}^+}{2(n+j-\alpha)+1} \\
a_{n,k-1}^+ = \delta_{0n}^- \left(\sum_{j=1}^{k} \frac{a_{j-1,k-j}^-}{2(n+j)} + \lambda^{1-2\alpha} \sum_{j=1}^{k} \frac{b_{j-1,k-j}}{2(n+j-\alpha)+1} \right) \\
(n = 0, 1, \ldots; k = 1, 2, \ldots)
\end{gathered}
\tag{2.14}
$$

As in [5], we can invert the recurrence relations (2.14) and obtain explicit formulas for the coefficients $a_{nk}^{\pm}$, b_{nk}. However, formulas (2.14) are more suitable for numerical calculations.

Let us now find the constant C and the position of the point b. Taking into account the condition of equilibrium of the stamp (2.4) and (2.5), we obtain $\Phi_1^-(0) = (2a)^{-1}P$, and this yields, finally, by virtue of (2.11),

$$
C = \frac{P}{2a\,\Gamma(\alpha)} \left(1 + \sum_{j=0}^{\infty} \frac{A_{j*}^+}{2(\alpha-1-j)}\right)^{-1}
$$

Just as in Sec. 1, we introduce the quantity

$$
K(b) = \lim_{r \to b-0} (b-r)^{1-\alpha} \chi_2(r)
$$

Taking into account (2.5) and remembering that it follows from (2.11) that

$$
\Phi_2^-(s) \sim \frac{s^{-\alpha}}{\varkappa_* 2^{1-\alpha}} C\Sigma, \quad s \to \infty, \quad s \in D^-; \ \Sigma = \sum_{j=0}^{\infty} (A_{j*}^- + B_{j*})
$$

we obtain $K(b) = C(\varkappa_* \Gamma(\alpha))^{-1}(b/2)^{1-\alpha}\Sigma$. Therefore the value of b can be obtained from the equation $\Sigma = 0$.

We will construct computational formulas for the contact stresses. Taking into account relations (2.3), (2.5) and (2.11) and using an inverse Mellin transformation, we obtain

$$
\begin{gathered}
\sigma_\theta(r, 0) = L_1(r) + \varkappa_- \varkappa_*^{-1} L_2(r) \\
L_1(r) = \frac{1}{2\pi i} \int_\Gamma \frac{C + \Psi_0^+(s)}{K_0^-(s)} \left(\frac{r}{a}\right)^{-s-1} ds, \\
L_2(r) = \frac{1}{2\pi i} \int_\Gamma \frac{\Psi_0^-(s) + \Psi_1^-(s)}{K_0^+(s)} \left(\frac{r}{b}\right)^{-s-1} ds
\end{gathered}
\tag{2.15}
$$

Using the theory of residues and the first equation of (2.12), we obtain

$$
\sigma_\theta(r, 0) = -\frac{\varkappa_-}{\varkappa_* \varkappa_0} \sum_{j=0}^{\infty} \frac{B_j \Gamma(3/2 + j - \alpha)}{\Gamma(j + 1/2)} \left(\frac{r}{b}\right)^{2j} \quad (0 < r < b)
$$

Consider the case when $b < r < a$. We have

$$
L_1(r) = \frac{(r/a)^{2\alpha-1}}{\Gamma(1-\alpha)} \left[2C\left(1 - \frac{r^2}{a^2}\right)^{-\alpha} - \sum_{m=0}^{\infty} \frac{A_m^+}{m+1} F\left(\alpha, m+1; m+2; \frac{r^2}{a^2}\right)\right]
$$

If $\max\{b, 2^{-1/2}a\} \leqslant r < a$, then we should use the following transformation formula [6] to calculate Gauss' function appearing in the last equation:

$$(m+1)^{-1} F(\alpha, m+1; m+2; r^2/a^2) = m! \, [(1-\alpha)_{m+1}]^{-1} (a/r)^{2m+2} +$$
$$+ (\alpha - 1)^{-1} (1 - r^2/a^2)^{1-\alpha} F(m+2-\alpha, 1; 2-\alpha; 1 - r^2/a^2)$$

Function $L_2(r)$ is given by the relation

$$L_2(r) = \frac{1}{\varkappa_0}\left(\frac{r}{b}\right)^{2\alpha-3} \sum_{m=0}^{\infty} \left\{A_m^- \left[\Gamma_1\left(\frac{1}{2}, \alpha; r\right) + \Gamma_2(\alpha; r)\right] + \right.$$
$$\left. + B_m [\Gamma_1(\alpha, {}^1/_2; r) + \Gamma_2({}^1/_2; r)]\right\} \quad (b < r < \min\{2^{1/2} b, a\})$$
$$\Gamma_1(\alpha_1, \alpha_2; r) = -\frac{\Gamma(m + {}^3/_2 - \alpha_1)}{\Gamma(m + \alpha_2)} \left(\frac{r}{b}\right)^{2m+3-2\alpha_1}$$
$$\Gamma_2(\alpha_1; r) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \frac{(m+\alpha_1)_j}{(\alpha)_j} \left(1 - \frac{b^2}{r^2}\right)^{j-1+\alpha}$$

and when $2^{1/2} b \leq r < a$, we have

$$L_2(r) = \frac{\sin \pi\alpha}{\pi \varkappa_0} \Gamma(2-\alpha) \left(\frac{r}{b}\right)^{2\alpha-3} \sum_{m=0}^{\infty} \left[A_m^- \Gamma_0\left(\frac{1}{2}; r\right) + B_m \Gamma_0(\alpha; r)\right]$$
$$\Gamma_0(\alpha_1; r) = (m + {}^3/_2 - \alpha_1)^{-1} F(2-\alpha, m + {}^3/_2 - \alpha_1; m + {}^5/_2 -$$
$$- \alpha_1; b^2/r^2)$$

From (2.1) it follows that for $b<r<a$ the shear stresses $\tau_{r\theta}(r, 0)$ can be expressed in terms of the normal stresses $\sigma_\theta(r, 0)$: $\tau_{r\theta}(r, 0) = -\mu\sigma_\theta(r, 0)$.

It remains to discuss the case of $0<r<b$. We have

$$\tau_{r\theta}(r, 0) = \frac{\sin \pi\alpha}{\pi \varkappa_*} \Gamma(2-\alpha) \frac{r}{b} \sum_{m=0}^{\infty} [A_m^- \Gamma_3(\alpha; r) + B_m \Gamma_3({}^1/_2; r)]$$
$$(0 < r < 2^{-1/2} b), \quad \Gamma_3(\alpha_1; r) = (1 - m - \alpha_1)^{-1} F(2-\alpha, 1-m-$$
$$- \alpha_1; 2 - m - \alpha_1; r^2/b^2)$$
$$\tau_{r\theta}(r, 0) = \frac{r}{\varkappa_* b} \sum_{m=0}^{\infty} [A_m^- \Gamma_4({}^1/_2; r) + B_m \Gamma_4(\alpha; r)] - \mu\sigma_\theta(r, 0)$$
$$(2^{-1/2} b \leq r < b), \quad \Gamma_4(\alpha_1; r) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \frac{(-m + \alpha_1 - {}^1/_2)_j}{(\alpha)_j} \left(1 - \frac{r^2}{b^2}\right)^{j-1+\alpha} \tag{2.16}$$

3. THE NON-SYMMETRIC CONTACT PROBLEM WITH FRICTION AND ADHESION

Let us consider the interaction between the elastic half-plane ($0<r<\infty$, $-\pi<\theta<0$) and the stamp ($0<r<a$, $\theta = 0$), to which are applied the vertical force P, the moment M and the horizontal force T (Fig. 2). The region of contact is split into the adhesion and friction zones. The boundary conditions of the problem are:

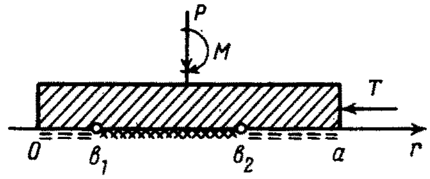

FIG. 2.

$$
\begin{gathered}
\theta=-\pi:\ \sigma_\theta=\tau_{r\theta}=0,\quad 0<r<\infty \\
\theta=0:\ u_\theta=\delta_n+\gamma r,\ 0<r<a;\ u_r=\delta_t,\ b_1<r<b_2 \\
\tau_{r\theta}-\mu\sigma_\theta=0,\ 0<r<b_1;\ \tau_{r\theta}+\mu\sigma_\theta=0,\ b_2<r<a \\
\tau_{r\theta}=\sigma_\theta=0,\quad a<r<\infty \\
|\tau_{r\theta}|<\mu|\sigma_\theta|,\quad b_1<r<b_2;\ \sigma_\theta>0,\quad 0<r<a
\end{gathered}
\tag{3.1}
$$

where γ is the angle of rotation of the stamp. The equilibrium of the stamp is ensured by the conditions

$$
\int_0^a \sigma_\theta(r,0)\,dr=P,\quad \int_0^a \tau_{r\theta}(r,0)\,dr=T,\quad \int_0^a \sigma_\theta(r,0)\,r dr=M \tag{3.2}
$$

We choose the following unknown functions:

$$
\begin{gathered}
\chi_1(r)=(\tau_{r\theta}-\mu\sigma_\theta)_{\theta=0},\quad \chi_2(r)=(\tau_{r\theta}+\mu\sigma_\theta)_{\theta=0} \\
\psi_1(r)=\nu_*^{-1}\partial u_r/\partial r\ (r,0),\quad \psi_2(r)=\nu_*^{-1}\partial u_\theta/\partial r\ (r,0)
\end{gathered}
\tag{3.3}
$$

and we have, by virtue of (3.1), $\operatorname{supp}\chi_1\subset[b_1,a]$, $\operatorname{supp}\chi_2\subset[0,b_2]$, $\operatorname{supp}\psi_1\subset[0,b_1]\cup[b_2,\infty)$, $\operatorname{supp}\psi_2\subset[0,\infty)$ when $0\leqslant r\leqslant a$ $\psi_2(r)=\nu_*^{-1}\gamma$). Writing

$$
\|\chi_{js},\psi_{js}\|=\int_0^\infty \|\chi_j(r),\psi_j(r)\|\,r^s dr
$$

we arrive, just as in Secs 1 and 2, at the relations

$$
\begin{gathered}
2\mu\psi_{1s}=l_{21}(s)\,\chi_{1s}+l_{22}(s)\,\chi_{2s},\quad 2\mu\psi_{2s}=l_{11}(s)\,\chi_{1s}+l_{12}(s)\,\chi_{2s} \\
l_{1j}(s)=-\mu\varkappa_-+(-1)^j\varkappa_+\operatorname{ctg}\pi s,\quad l_{2j}(s)=(-1)^j\varkappa_-+\mu\varkappa_+\operatorname{ctg}\pi s \\
(j=1,2)
\end{gathered}
\tag{3.4}
$$

Let us introduce the parameters

$$
\lambda_1=b_1/a,\quad \lambda_2=b_2/a,\quad 0<\lambda_1<\lambda_2<1
$$

and the functions

$$
\begin{gathered}
\Phi_1^-(s)=\int_{\lambda_1}^1 \chi_1(ar)\,r^s\,dr,\quad \Phi_1^+(s)=\int_1^{\lambda_1^{-1}} \chi_1(b_1 r)\,r^s\,dr \\
\Phi_2^-(s)=\int_0^1 \chi_2(b_2 r)\,r^s\,dr,\quad \Phi_2^+(s)=2\mu\int_1^\infty \psi_2(ar)\,r^s\,dr \\
\Phi_3^-(s)=2\mu\int_0^1 \psi_1(b_1 r)\,r^s\,dr,\quad \Phi_3^+(s)=2\mu\int_1^\infty \psi_1(b_2 r)\,r^s\,dr
\end{gathered}
\tag{3.5}
$$

Then, taking into account the fact that $\Phi_1^-(s) = \lambda_1^{s+1}\Phi_1^+(s)$ and using (3.4), we arrive at the Riemann vector problem

$$\begin{gathered}\Phi_1^+(s) = \lambda_1^{-s-1}\Phi_1^-(s)\\ \Phi_2^+(s) = l_{11}(s)\Phi_1^-(s) + \lambda_2^{s+1}l_{12}(s)\Phi_2^-(s) - (s+1)^{-1}C_0\\ \Phi_3^+(s) = \lambda_2^{-s-1}l_{21}(s)\Phi_1^-(s) + l_{22}(s)\Phi_2^-(s) - (\lambda_1/\lambda_2)^{s+1}\Phi_3^-(s)\\ s\in\Gamma,\ C_0 = 2\mu(1+\nu)^{-1}E\gamma\end{gathered} \tag{3.6}$$

We will obtain from the third equation of (3.6) first an expression for the function $\Phi_1^-(s)$ and then for $\Phi_2^-(s)$, and substitute them into the second equation, taking the first equation into account. As a result we obtain, instead of (3.6), the following system of functional equations:

$$\begin{gathered}C_0(s+1)^{-1} + \Phi_2^+(s) = l_{11}(s)\Phi_1^-(s) + \lambda_2^{s+1}l_{12}(s)\Phi_2^-(s)\\ \Phi_3^+(s) - \lambda_2^{-s-1}l_{11}^{-1}(s)\,l_{21}(s)\,[\Phi_2^+(s) + C_0(s+1)^{-1}] =\\ = l(s)\,l_{11}^{-1}(s)\,\Phi_2^-(s) - (\lambda_1/\lambda_2)^{s+1}\Phi_3^-(s)\\ -\Phi_3^-(s) = l(s)\,l_{12}^{-1}(s)\Phi_1^+(s) + (\lambda_2/\lambda_1)^{s+1}\Phi_3^+(s) - \lambda_1^{-s-1}l_{12}^{-1}(s)\times\\ \times l_{22}(s)\,[\Phi_2^+(s) + C_0(s+1)^{-1}],\ l(s) = -2\mu(\varkappa_-^2 + \varkappa_+^2\operatorname{ctg}^2\pi s)\end{gathered} \tag{3.7}$$

Let us factorize $l_{11}, l_{11}^{-1}l, l_{12}^{-1}l$:

$$l_{11}(s) = L_0^+(s)\,L_0^-(s),\quad l_{1j}^{-1}(s)\,l(s) = L_j^+(s)\,L_j^-(s)\quad (j=1,\ 2)$$

$$L_0^+(s) = \frac{\varkappa_0\Gamma(-s)}{\Gamma(1-\alpha-s)},\quad L_0^-(s) = \frac{\Gamma(1+s)}{\Gamma(\alpha+s)}$$

$$L_1^+(s) = -\frac{\varkappa_1\Gamma(-s)\Gamma(1-\alpha-s)}{\Gamma(^1/_2-i\beta-s)\Gamma(^1/_2+i\beta-s)},\quad L_1^-(s) = \frac{\Gamma(1+s)\Gamma(\alpha+s)}{\Gamma(^1/_2+i\beta+s)\Gamma(^1/_2-i\beta+s)}$$

$$L_2^+(s) = \frac{\varkappa_1\Gamma(-s)\Gamma(\alpha-s)}{\Gamma(^1/_2-i\beta-s)\Gamma(^1/_2+i\beta-s)},\quad L_2^-(s) = \frac{\Gamma(1+s)\Gamma(1-\alpha+s)}{\Gamma(^1/_2+i\beta+s)\Gamma(^1/_2-i\beta+s)}$$

$$\varkappa_1 = 2\mu\varkappa\varkappa_0^{-1},\quad \varkappa_0 = \varkappa_+\operatorname{cosec}\pi\alpha$$

The quantities α, β are given in (1.5). Just as in Sec. 2, we introduce the functions

$$\begin{gathered}\Psi_1^+(s) = \sum_{j=0}^{\infty}\frac{A_j^+}{s-1+\alpha-j},\quad \Psi_1^-(s) = \sum_{j=0}^{\infty}\frac{A_j^-}{s+\alpha+j}\\ \Psi_2^+(s) = \sum_{j=0}^{\infty}\frac{B_j^+}{s-s_j},\quad \Psi_2^-(s) = \sum_{j=0}^{\infty}\frac{B_j^-}{s+s_j}\\ s_{2m} = m + i\beta + {}^1/_2,\ s_{2m+1} = m - i\beta + {}^1/_2\ (m = 0, 1, \ldots)\end{gathered} \tag{3.8}$$

The coefficients $A_j^\pm$, $B_j^\pm$ are to be determined. Let us rewrite system (3.7) in the form

$$\begin{gathered}[L_0^+(s)]^{-1}[\Phi_2^+(s) + C_0(s+1)^{-1}] - \nu_0C_0(s+1)^{-1} - \Psi_1^+(s) =\\ = L_0^-(s)\,\Phi_1^-(s) + \lambda_2^{s+1}[L_0^+(s)]^{-1}l_{12}(s)\,\Phi_2^-(s) - \nu_0C_0(s+1)^{-1} -\\ - \Psi_1^+(s)\ (=C_1)\\ [L_1^+(s)]^{-1}\Phi_3^+(s) - \lambda_2^{-s-1}l_{21}(s)\,[l_{11}(s)\,L_1^+(s)]^{-1}[\Phi_2^+(s) +\\ + C_0(s+1)^{-1}] + \nu_1C_0(s+1)^{-1} - \Psi_1^-(s) - \Psi_2^+(s) = L_1^-(s)\,\Phi_2^-(s) -\\ - (\lambda_1/\lambda_2)^{s+1}[L_1^+(s)]^{-1}\Phi_3^-(s) + \nu_1C_0(s+1)^{-1} - \Psi_1^-(s) - \Psi_2^+(s)\ (=C_2)\end{gathered} \tag{3.9}$$

$$
\begin{gathered}
-[L_2^-(s)]^{-1}\Phi_3^-(s)-\Psi_2^-(s)=L_2^+(s)\,\Phi_1^+(s)+\\
+(\lambda_1/\lambda_2)^{-s-1}[L_2^-(s)]^{-1}\Phi_3^+(s)-\\
-\lambda_1^{-s-1}l_{22}(s)\,[l_{12}(s)\,L_2^-(s)]^{-1}[\Phi_2^+(s)\quad+C_0(s+1)^{-1}]-\\
-\Psi_2^-(s)\ (=0)\\
\nu_0=\varkappa_0^{-1}\Gamma(2-\alpha),\ \nu_1=\Gamma(\alpha)\,(\beta^2+{}^1/_4)\,[2\,(1-\alpha)\,\varkappa^{s/s}]^{-1}
\end{gathered}
$$

(C_1, C_2 are arbitrary constants). In order to satisfy Liouville's theorem used here, it is necessary and sufficient that the coefficients

$$
A_n^{\pm}=\sum_{k=0}^{2}C_kA_{nk}^{\pm},\qquad B_n^{\pm}=\sum_{k=0}^{2}C_kB_{nk}^{\pm} \tag{3.10}
$$

are the solution of the following infinite, normal-type algebraic system:

$$
\begin{gathered}
A_{nk}^-=\lambda_2^{\alpha+n-1}r_n^{(1)}\left(\frac{\nu_0\delta_{k0}}{n+\alpha-1}-\delta_{k1}+\sum_{j=0}^{\infty}\frac{A_{jk}^+}{n+j+1}\right)\\
B_{nk}^+=\left(\frac{\lambda_1}{\lambda_2}\right)^{s_n+1}r_n^{(3)}\sum_{j=0}^{\infty}\frac{B_{jk}^-}{s_n+s_j}\\
A_{nk}^+=\lambda_2^{2-\alpha+n}r_n^{(2)}\left[-\frac{\nu_1\delta_{k0}}{2-\alpha+n}+\delta_{k2}+\sum_{j=0}^{\infty}\left(\frac{A_{jk}^-}{n+j+1}+\frac{B_{jk}^+}{n+1-\alpha-s_j}\right)\right]\\
B_{nk}^-=\left(\frac{\lambda_1}{\lambda_2}\right)^{s_n-1}r_n^{(4)}\left[\frac{\nu_1\delta_{k0}}{s_n-1}+\delta_{k2}+\sum_{j=0}^{\infty}\left(\frac{A_{jk}^-}{-s_n+\alpha+j}-\frac{B_{jk}^+}{s_n+s_j}\right)\right]\\
n=0,\,1,\,\ldots;\ k=0,\,1,\,2
\end{gathered} \tag{3.11}
$$

$$
\begin{gathered}
r_n^{(1)}=\varkappa_-\sin\pi\alpha\,(\mu^2+1)\,(\pi\varkappa_1 n!^{\,2})^{-1}\Gamma({}^1/_2+\bar z+n)\,\Gamma({}^1/_2+z+n)\\
r_n^{(2)}=-\sin 2\pi\alpha\,(\pi n!^2)^{-1}\,\Gamma({}^3/_2-\bar z+n)\,\Gamma({}^3/_2-z+n),\ z=\alpha+i\beta\\
r_{2m}^{(3)}=\varkappa_1^{-1}R_mL_2^-(s_{2m}),\ r_{2m}^{(4)}=-(s_{2m})^{-1}R_mL_1^+(-s_{2m})\\
R_m=(-1)^m\,\Gamma(-m-2i\beta)\,[m!\,\Gamma(-s_{2m})\,\Gamma(1-\alpha-s_{2m})]^{-1},\\
r_{2m+1}^{(k)}=\overline{r_{2m}^{(k)}}\ (k=3,\,4)
\end{gathered}
$$

(δ_{km} is the Kronecker delta). We find the solution of the Riemann problem (3.6) from (3.9)

$$
\begin{gathered}
\Phi_1^-(s)=[L_0^-(s)]^{-1}\Omega_1(s)+\lambda_1^{s+1}[L_2^+(s)]^{-1}\Psi_2^-(s)-\\
-\lambda_2^{s+1}L_1^+(s)\,[L_2^+(s)\,L_2^-(s)]^{-1}\Omega_2(s),\ \Phi_1^+(s)=\lambda_1^{-s-1}\Phi_1^-(s)\\
\Phi_2^-(s)=[L_1^-(s)]^{-1}\Omega_2(s)-(\lambda_1/\lambda_2)^{s+1}L_2^-(s)[L_1^+(s)L_1^-(s)]^{-1}\Psi_2^-(s)\\
\Phi_2^+(s)=-C_0(s+1)^{-1}+L_0^+(s)\,\Omega_1(s),\ \ \Phi_3^-(s)=-L_2^-(s)\,\Psi_2^-(s)\\
\Phi_3^+(s)=L_1^+(s)\,\Omega_2(s)+\lambda_2^{-s-1}l_{21}(s)\,l_{11}^{-1}(s)\,L_0^+(s)\,\Omega_1(s)\\
\Omega_1(s)=C_0\nu_0\,(s+1)^{-1}+C_1+\Psi_1^+(s)\\
\Omega_2(s)=-C_0\nu_1\,(s+1)^{-1}+C_2+\Psi_1^-(s)+\Psi_2^+(s)
\end{gathered} \tag{3.12}
$$

Next we determine the angle of rotation γ and the constants C_1 and C_2. Taking into account the notation (3.3) and (3.5), we can write the conditions of equilibrium of the stamp (3.2) in the form

$$\lambda_2\Phi_2^-(0) - \Phi_1^-(0) = 2\mu a^{-1}P,\quad \lambda_2\Phi_2^-(0) + \Phi_1^-(0) = 2a^{-1}T$$
$$\lambda_2^2\Phi_2^-(1) - \Phi_1^-(1) = 2\mu a^{-2}M \tag{3.13}$$

Taking into account formulas (3.12) and (3.8) and writing

$$\omega_{km}^{\pm} = \sum_{j=0}^{\infty}\frac{A_{jk}^{\pm}}{\alpha \mp j + m - {}^1/_2 \mp {}^1/_2},\quad \omega_{km} = \sum_{j=0}^{\infty}\frac{B_{jk}^{+}}{m - s_j}$$
$$d_0 = \pi\,[\Gamma(\alpha)\,\mathrm{ch}\,\pi\beta]^{-1},\; d_1 = \pi\,(\beta^2 + {}^1/_4)\,[\Gamma(\alpha+1)\,\mathrm{ch}\,\pi\beta]^{-1}$$

we obtain the following system of three equations for determining the constants C_0, C_1, C_2 from conditions (3.13):

$$\sum_{k=0}^{2} a_{jk}C_k = f_j \quad (j = 0, 1, 2);\quad f_0 = 2\mu\frac{P}{a},\quad f_1 = 2\frac{T}{a},\quad f_2 = 2\mu\frac{M}{a^2} \tag{3.14}$$
$$a_{jk} = 2\lambda_2 d_0\delta_{j1}(-\nu_1\delta_{k0} + \delta_{k2} + \omega_{k0}^- + \omega_{k0}) - (-1)^j\Gamma(\alpha)\times$$
$$\times(\nu_0\delta_{k0} + \delta_{k1} + \omega_{k0}^+)\quad (j = 0, 1)$$
$$a_{2k} = -\Gamma(\alpha+1)\,({}^1/_2\nu_0\delta_{k0} + \delta_{k1} + \omega_{k1}^+)\quad (k = 0, 1, 2)$$

The angle of rotation γ is connected with C_0 by the relation (3.6).

In order to find the unknown points b_1, b_2 at which the boundary conditions change, we introduce, as in Secs 1 and 2, the stress intensity factors

$$K_1 = \lim_{r\to b_1+0}(r - b_1)^{1-\alpha}\chi_1(r),\; K_2 = \lim_{r\to b_2-0}(b_2 - r)^{1-\alpha}\chi_2(r) \tag{3.15}$$

On the one hand we have, by virtue of (3.12), as $s\to\infty$

$$\Phi_1^+(s) \sim \varkappa_1^{-1}\omega_*(-s)^{-\alpha}\quad (s\in D^+),\quad \Phi_2^-(s) \sim C_2 s^{-\alpha}\quad (s\in D^-)$$
$$\omega_* = B_0^- + B_1^- + \ldots \tag{3.16}$$

while on the other we have, from (3.5) and (3.15), as $s\to\infty$

$$\Phi_1^+(s) \sim K_1 b_1^{\alpha-1}\Gamma(\alpha)(-s)^{-\alpha}\quad (s\in D^+),\quad \Phi_2^-(s) \sim K_2 b_2^{\alpha-1}\Gamma(\alpha)\,s^{-\alpha}$$
$$(s\in D^-)$$

and we find

$$K_1 = b_1^{1-\alpha}[\varkappa_1\Gamma(\alpha)]^{-1}\omega_*,\; K_2 = b_2^{1-\alpha}[\Gamma(\alpha)]^{-1}C_2$$

The parameters λ_1, λ_2 are found from the conditions $K_1 = 0$, $K_2 = 0$, which are obviously equivalent to the relations $C_2 = 0$, $\omega_* = 0$. Taking into account (3.16) and (3.10), we reduce the linear system of three equations (3.14) for C_0, C_1, C_2 to the system of two transcendental equations for λ_1, λ_2

$$f_0(a_{j0}F_1 - a_{j1}F_0) = (a_{00}F_1 - a_{01}F_0)\,f_j\quad (j = 1, 2)$$
$$F_k = B_{0k}^- + B_{1k}^- + \ldots\quad (k = 0, 1)$$

Here the angle of rotation is given by the equation

$$\gamma = (1 + \nu) f_0 [2\mu E (a_{00} - F_1^{-1} F_0 a_{01})]^{-1}$$

and the equilibrium of the stamp is ensured.

Let us now investigate the problem of singularities in contact stresses and displacements in the neighbourhood of the singularities. Since $K_1 = K_2 = 0$, the functions $\tau_{r\theta}$, σ_θ, $\partial u_r/\partial r$, $\partial u_\theta/\partial r$ are bounded when $\theta = 0$ and $r \to b_1$, $r \to b_2$. In the neighbourhood of the point $r = a$ we have

$$\sigma_\theta (r, 0) = O\{(a - r)^{-\alpha}\}, \quad \tau_{r\theta} (r, 0) = O\{(a - r)^{-\alpha}\}, \quad r \to a - 0$$
$$\partial u_\theta/\partial r (r, 0) = O\{(r - a)^{-\alpha}\}, \quad \partial u_r/\partial r (r, 0) = O(1), \quad r \to a + 0$$

The study of the behaviour of the stresses of $r \to 0$ produces non-trivial results. We have, by virtue of (3.5) and (3.12),

$$\chi_2 (r) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{\Omega_2 (s)}{L_1^- (s)} - \left(\frac{\lambda_1}{\lambda_2}\right)^{s+1} \frac{L_2^- (s) \Psi_2^- (s)}{L_1^+ (s) L_1^- (s)} \right] \left(\frac{r}{b_2}\right)^{-s-1} ds \tag{3.17}$$

The function $[L_1^-(s)]^{-1}$ has, in the region D^+, poles at the points $s = -s_n$ [s_n are complex numbers defined in (3.8)]. However, by virtue of (3.12) and the fourth relation of (3.11), we have

$$\operatorname*{Res}_{s=-s_n} \{[\Omega_2 (s) - (\lambda_1/\lambda_2)^{s+1} L_2^- (s) \Psi_2^- (s) (L_1^+ (s))^{-1}] [L_1^- (s)]^{-1}\} = 0$$

Applying the theory of residues and taking into account the last equation, we obtain from (3.17) and (3.3)

$$\chi_2 (r) = -\Psi_2^- (\alpha - 1) [\Gamma (2\alpha - 1) L_1^+ (\alpha - 1)]^{-1} (r/b_1)^{-\alpha} + O(r^{1-\alpha}), \quad r \to 0$$
$$\tau_{r\theta} (r, 0) = {}^1/_2 \chi_2 (r), \quad \tau_{r\theta} (r, 0) = (2\mu)^{-1} \chi_2 (r), \quad 0 < r < b_1$$

Comparing (3.5) and (3.12), we obtain

$$\left.\frac{\partial u_r}{\partial r}\right|_{\theta=0} = -\frac{1}{2\mu} \frac{\Gamma (\alpha) \Psi_2^- (\alpha - 1) (r/b_1)^{-\alpha}}{\Gamma (-{}^1/_2 + i\beta + \alpha) \Gamma (-{}^1/_2 - i\beta + \alpha)} + O(1), \quad r \to 0$$

In conclusion we note the possibility of a passage to the limit $\lambda_1 \to 0$ ($b_1 \to 0$) in formulas (3.12). In this case we have $\Phi_1^+(s) = 0$, $\Phi_3^-(s) = 0$, $\Psi_2^\pm(s) = 0$. To obtain the coefficients $A_n^\pm$ we turn to the algebraic system

$$A_n^- = -\lambda_2^{-1+\alpha+n} r_n^{(1)} \left(C_1 + \frac{C_0 \nu_0}{1 - \alpha - n} - \sum_{j=0}^{\infty} \frac{A_j^+}{1 + n + j} \right)$$
$$A_n^+ = \lambda_2^{2-\alpha+n} r_n^{(2)} \left(C_2 - \frac{C_0 \nu_1}{2 - \alpha + n} + \sum_{j=0}^{\infty} \frac{A_j^-}{1 + n + j} \right)$$

Taking into account relation (3.12) ($\Psi_2^\pm = 0$), we obtain (D_j are complex constants)

$$\chi_j (r) = D_j r^{-{}^1/_2 + i\beta} + \bar{D}_j r^{-{}^1/_2 - i\beta} + O(r^{{}^1/_2}), \quad r \to 0 \quad (j = 1, 2)$$

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THE CONTACT PROBLEM OF THE DISCRETE FITTING OF AN INHOMOGENEOUS VISCOELASTIC AGEING CYLINDER WITH A SYSTEM OF RIGID COLLARS†

A. V. MANZHIROV and V. A. CHERNYSH

Moscow

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The axially symmetric contact problem of the interaction of an inhomogeneous ageing viscoelastic cylindrical body with an arbitrary finite system of fitted rigid elements is considered. Account is taken of the fact that the collars are not fitted or removed at the same time, which is dictated, for example, by the particular features of the installation of engineering structures, as well as the properties of the age and structural inhomogeneities of the deforming body itself due to manufacturing processes or the erection of real objects. A formulation of the problem and its system of resolvent bidimensional integral equations are given. A solution of the system is constructed. A numerical analysis of a number of actual processes is carried out and the mechanisms of both the individual as well as the combined effect of the main factors on the characteristics of the contact interaction are investigated.

1. FORMULATION AND RESOLVENT EQUATIONS OF THE CONTACT PROBLEM

LET US investigate the process of the sequential fitting of rigid collars to a bilayer hollow cylinder, the layers of which are made out of different viscoelastic ageing materials at different instants of

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